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PROPERTIES WHICH NORMAL OPERATORS SHARE WITH NORMAL DERIVATIONS AND RELATED OPERATORS

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Let S and T be (bounded) scalar operators on a Banach space $\mathscr{X}$ and let $C(T, S)$ be the map on $\mathscr{B}(\mathscr{X})$, the bounded linear operators on $\mathscr{X}$, defined by

$$C(T, S)(X) = TX - XS$$

for X in $\mathscr{B}(\mathscr{X})$. This paper was motivated by the question: to what extent does $C(T, S)$ behave like a normal operator on Hilbert space? It will be shown that $C(T, S)$ does share many of the special properties enjoyed by normal operators. For example it will be shown that the range of $C(T, S)$ meets its null space at a positive angle and that $C(T, S)$ is Hermitian if T and S are Hermitian. However, if $\mathscr{X}$ is a Hilbert space then $C(T, S)$ is a spectral operator if and only if the spectrum of T and the spectrum of S are both finite.

We now indicate our results in greater detail. Let $\mathscr{H}$ be a Hilbert space and let N be a normal operator in $\mathscr{B}(\mathscr{H})$. Then N enjoys the following properties.

(A) $\mathscr{R}(N)$ is orthogonal to $\mathscr{N}(N)$, where $\mathscr{R}(N)(\mathscr{N}(N))$ donotes the range (null space) of N.

(B) $\mathscr{R}(N)^- \oplus \mathscr{N}(N) = \mathscr{H}$, where the bar denotes norm closure.

(C) There is a resolution of the identity $E(\cdot)$ supported by $\sigma(N)$ such that

$$N = \int_{\sigma(N)} \lambda dE ,$$

where $\sigma(N)$ denotes the spectrum of N. That is, N is a scalar operator.

(D) If $x \in E(\{\lambda\})\mathscr{H}$ for some complex number λ, then $Nx = \lambda x$.

(E) N has closed range if and only if 0 is an isolated point in $\sigma(N)$. (We adopt the convention that 0 is isolated in $\sigma(N)$ if $0 \notin \sigma(N)$).

(F) The norm, spectral radius, and numerical radius of N are equal.

(G) The closure of the numerical range of N is the convex hull of the spectrum of N.

In §§ 1, 2, and 3 of this paper we show that appropriate versions of (A), (D), and (E) hold for $C(T, S)$. In Section 4 we restrict ourselves to the Hilbert space case and show that (B) is false in general. This result enables us to give the characterization of operators of the form $C(T, S)$ which are spectral operators which was mentioned above. In § 5 assuming that T and S are normal operators on a Hilbert space, we show that (G) holds but that in general (F) is false for $C(T, S)$. We conclude the paper with an example of a Hermitian operator whose square is not Hermitian.

In the sequel an operator shall be called spectral (scalar) if it is spectral (scalar) in the sense of Dunford. An operator shall be called Hermitian if it is Hermitian in the sense of Lumer and Vidav (see [7]). We shall make use of the theory of decomposable operators as presented in [3]. If T is a decomposable operator on a Banach space $\mathscr{X}$ and F is a closed subset of the complex plane $\boldsymbol{C}$ (or the real line $\boldsymbol{R}$) we shall usually denote the spectral maximal subspace of $\mathscr{X}$ associated with T and F by $\mathscr{X}_T(F)$. However, the spectral maximal subspace of $\mathscr{B}(\mathscr{X})$ which is associated with $C(T, S)$ and the complex set F shall be denoted by $\mathscr{B}_C(F)$. The derivation $C(T, T)$ shall sometimes also be written as Δ_T. Following [1], if N is a normal operator on a Hilbert space we shall call Δ_N the normal derivation induced by N.

1. It is shown in [1] that if N is a normal operator in $\mathscr{B}(\mathscr{H})$ then

$$\| Y - \Delta_N(X) \| \leqq \| Y \|$$

for all X in $\mathscr{B}(\mathscr{H})$ and all Y in $\mathscr{N}(\Delta_N)$. In this section we obtain a generalized inequality for $C(T, S)$. Since the proofs are generally similar to those given in [1], we will be brief.

Definition 1.1. Let $\mathscr{M}$ and $\mathscr{N}$ be (not necessarily closed) subspaces of a normed linear space $\mathscr{X}$. We shall say that *$\mathscr{M}$ meets $\mathscr{N}$ at angle θ* $(0 \leqq \theta \leqq \pi/2)$ where by definition

$$\sin \theta = \inf \{\| m + n \| : m \in \mathscr{M},\ n \in \mathscr{N};\ \| n \| = 1\} .$$

If $\mathscr{M}$ meets $\mathscr{N}$ at angle $\pi/2$ we say that *$\mathscr{M}$ is orthogonal to $\mathscr{N}$*. It is easy to show that $\mathscr{M}$ meets $\mathscr{N}$ at angle 0 if and only if $\mathscr{N}$ meets $\mathscr{M}$ at angle 0 so that if $\mathscr{M}$ meets $\mathscr{N}$ at angle $\alpha > 0$ then $\mathscr{N}$ meets $\mathscr{M}$ at angle $\beta > 0$. In general, however, α need not equal β.

If T is an invertible element of $\mathscr{B}(\mathscr{X})$ then T is said to be *power bounded by K* if for some $K \geqq 1$, $\| T^n \| \leqq K$ for $n = \pm 1, \pm 2, \cdots$.

Lemma 1.2. *If T and S are invertible elements of $\mathscr{B}(\mathscr{X})$*

which are both power bounded by K *then* $\mathscr{R}(C(T, S))$ *meets* $\mathscr{N}(C(T, S))$ *at angle* θ, *where* $\sin\theta \geqq 1/K^2$.

LEMMA 1.3. *If* S *and* T *are in* $\mathscr{B}(\mathscr{X})$ *and* $i \notin \sigma(T) \cup \sigma(S)$ *then* $\mathscr{R}(C(T, S)) = \mathscr{R}(C(W, V))$ *where* $W = (T - iI)(T + iI)^{-1}$ *and* $V = (S - iI)(S + iI)^{-1}$ *are the Cayley transforms of* T *and* S *respectively.*

LEMMA 1.4. *Let* T *and* S *be scalar elements of* $\mathscr{B}(\mathscr{X})$ *and let* f *and* g *be real-valued Borel measurable functions on* $\mathbf{C}$ *which assume only finitely many* (real) *values. Then the Cayley transforms of* $f(T)$ *and* $g(S)$ *are both power bounded by* K *for some* $K > 0$. *Furthermore, the constant* K *does not depend on the particular choice of* f *or* g.

The proof of (1.2) depends on the following generalization of the identity used to prove (1.4) in [1]: If X and Y are in $\mathscr{B}(\mathscr{X})$ and $TY = YS$ then for each integer n

$$nT^{n-1}Y = T^nX - XS^n - \sum_{k=0}^{n-1} T^{n-k-1}(TX - XS - Y)S^k .$$

The proof of (1.3) is an obvious modification of the proof of (1.5) in [1]. (1.4) follows easily from [5, Theorem 7, p. 330] (the constant K depends only on the norms of the spectral projections associated with T and S). Note that if T and S are normal operators on Hilbert space K may be taken to be 1 in (1.4).

THEOREM 1.5. *If* S *and* T *are scalar operators then there is a real number* $\theta > 0$ *such that the range of* $C(T, S)$ *meets the null space of* $C(T, S)$ *at angle* θ. *If* T *and* S *are normal operators on a complex Hilbert space then* $\mathscr{R}(C(T, S))$ *is orthogonal to* $\mathscr{N}(C(T, S))$.

Proof. Let $E(\cdot)$ and $F(\cdot)$ be the spectral resolutions of identity associated with T and S respectively. Partition $\sigma(T) \cup \sigma(S)$ into rectangles $\delta_1, \delta_2, \cdots, \delta_n$ and let λ_i be the midpoint of δ_i for $i = 1, 2, \cdots, n$. Let X and Y be in $\mathscr{B}(\mathscr{X})$ and suppose Y is in $\mathscr{N}(C(S, T))$. Consider

$$(*) \qquad \left\| Y - \sum_{k=1}^{n} \lambda_k E(\delta_k)X - X\sum_{k=1}^{n} \lambda_k F(\delta_k) \right\| .$$

To prove the theorem, it suffices to show that $(*) \geqq \sin\theta\, \|Y\|$ for some $\theta > 0$ and all possible partitions of $\sigma(T) \cup \sigma(S)$. As in [1, (1.6)], a computation shows that the range of the map $X \mapsto \sum_{k=1}^{n} \lambda_k E(\delta_k)X - X\sum_{k=1}^{n} \lambda_k F(\delta_k)$ does not change if the λ_k's are replaced by any complex n-tuple $\{\mu_1, \mu_2, \cdots, \mu_n\}$ where the μ_k's are distinct. Hence, in $(*)$ we may replace X by a suitable X_1 and meanwhile assume that λ_k is real for $k = 1, 2, \cdots, n$. Then, by (1.3) we may replace X_1 by another suitable X_2 and replace $\sum_{k=1}^{n} \lambda_k E(\delta_k)$ and $\sum_{k=1}^{n} \lambda_k F(\delta_k)$ by their Cayley transforms. By (1.4) these Cayley transforms are power bounded by K. Hence, our first assertion follows from (1.2) with $\sin\theta \geqq 1/K^2$. If T and S are normal operators the constant K in (1.2) and (1.4) may be taken to be 1. It follows that $\theta = \pi/2$.

COROLLARY 1.6. *If N is a normal operator in $\mathscr{B}(\mathscr{H})$ and λ_1 and λ_2 are distinct eigenvalues of Δ_N with corresponding eigenspaces $\mathscr{X}_1$ and $\mathscr{X}_2$, then $\mathscr{X}_1$ is orthogonal to $\mathscr{X}_2$ and $\mathscr{X}_2$ is orthogonal to $\mathscr{X}_1$.*

Proof. $\mathscr{X}_1 = \mathscr{N}(\Delta_N - \lambda_1 I) - \mathscr{N}(C(N, N) - \lambda_1 I) = \mathscr{N}(C(N - \lambda_1 I, N))$ (I denotes either the identity on $\mathscr{H}$ or the identity on $\mathscr{B}(\mathscr{H})$). If $\Delta_N(X) = \lambda_2 X$ then $C(N - \lambda_1 I, N)(X) = (\lambda_2 - \lambda_1)X$ so X is in the range of $C(N - \lambda_1 I, N)$. Hence, by (1.5) $\mathscr{X}_2$ is orthogonal to $\mathscr{X}_1$. Similarly, $\mathscr{X}_1$ is orthogonal to $\mathscr{X}_2$.

2. If S and T are scalar operators on $\mathscr{X}$, then [3, p. 112] $C(T, S)$ is a generalized scalar operator and, hence, a decomposable operator. Thus, $\mathscr{B}_C(\{\lambda\})$ is a spectral maximal subspace for each λ in the spectrum of $C(T, S)$. In this section we prove that $\mathscr{B}_C(\{\lambda\})$ consists solely of eigenvectors (i.e. (D) holds for $C(T, S)$) and give two examples.

THEOREM 2.1. *$\mathscr{B}_C(\{\lambda\}) = \mathscr{N}(C - \lambda I)$ for each λ in $\sigma(C(T, S))$, if T and S are scalar operators in $\mathscr{B}(\mathscr{X})$.*

Proof. Since $T - \lambda I$ is a scalar operator if T is a scalar operator we may assume $\lambda = 0$. Let $E(\cdot)$ and $F(\cdot)$ be the spectral resolutions of the identity associated with T and S respectively. Suppose that $X \in \mathscr{B}(\mathscr{X})$ and that $\| C^n(T, S)(X) \|^{1/n} \to 0$ as $n \to \infty$. Then [3, 4.5, p. 113] $E(\delta)XF(\delta) = XF(\delta)$ for each closed subset δ of $\boldsymbol{C}$. Hence,

$$E(\delta)XF(\gamma) = E(\delta \cap \gamma)XF(\gamma) = 0$$

for all Borel subsets γ with closure disjoint from δ. Let δ be a closed subset of $\boldsymbol{C}$ and let $\{\gamma_k\}$ be disjoint Borel sets which cover the complement of δ and have closures from δ. Then

$$\begin{aligned} E(\delta)X &= E(\delta)XF(U\gamma_k) + E(\delta)XF(\delta) \\ &= E(\delta)X\Big(\sum_{k=1}^{\infty} F(\gamma_k)\Big) + E(\delta)XF(\delta) \\ &= E(\delta)XF(\delta)\,. \end{aligned}$$

Therefore, $E(\delta)X = XF(\delta)$. Since Borel measures are regular and $E(\cdot)$ and $F(\cdot)$ are countably additive, $E(\delta)X = XF(\delta)$ for each Borel set $\delta \subset C$ and so $TX = XS$, i.e., $C(T, S)(X) = 0$. The theorem now follows from [3, 4.4, p. 113].

EXAMPLES Let $\tau = \{\lambda \in C: |\lambda| = 1\}$ and let U be multiplication by λ on $L^2(\tau)$. Thus, U is the simple bilateral shift. Let $E(\cdot)$ be the spectral measure associated with U. We show that Δ_U has no non-zero eigenvalues. For if $\Delta_U(X) = -\lambda X$ with $\lambda \neq 0$ then as in (2.1) $E(\delta)X = XE(\delta + \lambda)$ for each Borel set $\delta(\delta + \lambda$ is the translate of δ by $\lambda)$. Therefore,

$$X = E(\tau)XE(\tau) = XE((\tau + \lambda) \cap \tau)\,.$$

But $(\tau + \lambda) \cap \tau$ consists of at most two points and $E(\{\mu\}) = 0$ for each μ in C; thus $X = 0$.

On the other hand, if M is multiplication by x on $L^2(0, 1)$ and W_a is the operator defined on $L^2(0, 1)$ by

$$(W_a f)(x) = \int_a^1 f(x - t)d\mu_a(t) = \begin{cases} f(x - a) & \text{if } a \leqq x \leqq 1 \\ 0 & \text{if } x < a \end{cases}$$

where $a \in (0, 1)$ is fixed, then a simple computation shows

$$MW_a - W_aM = aW_a\,.$$

Thus, depending (in part) on the geometry of the spectrum of T, $\mathscr{B}_{\Delta_T}(\{\lambda\})$ may or may not be empty.

3. Our goal in this section is to show that (E) holds for $C(T, S)$.

THEOREM 3.1. *If T is a decomposable operator in $\mathscr{B}(\mathscr{X})$ such that the range of T is closed and the range of T meets the null space of T at angle $\theta > 0$, then 0 is an isolated point of the spectrum of T.*

Proof. Since $\mathscr{R}(T)$ is closed, by the open mapping theorem there is a constant $M > 0$ such that each y in $\mathscr{R}(T)$ has the form $y = Tx$ where $x \in \mathscr{X}$ and $\|x\| \leqq M\|y\|$. Also, since $\mathscr{R}(T)$ meets $\mathscr{N}(T)$ at a positive angle, $\mathscr{N}(T)$ meets $\mathscr{R}(T)$ at angle $\varphi > 0$.

For each $r > 0$ let $F_r = \{\lambda \in C: |\lambda| \geqq r\}$ and let $T_r = T|_{\mathscr{X}_T(F_r)}$. Then $\sigma(T_r) \subseteqq F_r$ so T_r^{-1} exists and is bounded. Let $1/r_0 = M/\sin\varphi$. We show that $\| T_r^{-1} \| \leqq 1/r_0$ for all $r > 0$. If $y \in \mathscr{X}_T(F_r)$ then $y \in \mathscr{R}(T)$ so $y = Tx$ for some $x \in \mathscr{X}$ with $\|x\| \leqq M\|y\|$. Let $w = x - T_r^{-1}y$. Then $Tw = 0$ so, since $\mathscr{N}(T)$ meets $\mathscr{R}(T)$ at angle φ

$$\sin\varphi \| T_r^{-1}y \| \leqq \| w + T_r^{-1}y \| = \|x\| \leqq M\|y\| .$$

Since y was arbitrary, $\| T_r^{-1} \| \leqq M/\sin\varphi = 1/r_0$. It follows that $\sigma(T_r) \subset F_{r_0}$ for all $r > 0$. Indeed, if $T_r - \lambda I$ were not invertible for some λ in C with $0 < |\lambda| < r_0$ then because T_r is a decomposable operator, $\|(T_r - \lambda I)x_n\| \to 0$ for a sequence x_n of unit vectors in $\mathscr{X}_T(F_r)$ and

$$\begin{aligned} |1/|\lambda| - \| T_r^{-1}x_n \|| &\leqq (1/|\lambda|)\| x_n - \lambda T_r^{-1}x_n \| \\ &\leqq (\| T_r^{-1} \|/|\lambda|)\|(T_r - \lambda I)x_n\| \end{aligned}$$

so that $\| T_r^{-1} \| \geqq 1/|\lambda| > 1/r_0$, a contradiction. Thus, $\sigma(T) \subset F_{r_0} \cup \{\lambda \in C: |\lambda| < r\}$ for all $r > 0$ and so 0 is an isolated point of $\sigma(T)$.

EXAMPLE 3.2. Let V be the Volterra operator defined on $L^2(0, 1)$ by

$$(Vf)(x) = \int_0^x f(t)dt .$$

Then V is an injective compact quasinilpotent operator with dense range. Hence, V is a decomposable operator such that 0 is an isolated point of its spectrum and $\mathscr{R}(V)$ is orthogonal to $\mathscr{N}(V)(= \{0\})$. However, since V is a compact operator, its range cannot be closed.

Recall that Lumer and Rosenblum have shown [6] that for any T and S in $\mathscr{B}(\mathscr{X})$

$$\sigma(C(T, S)) = \sigma(T) - \sigma(S) = \{\lambda - \mu: \lambda \in \sigma(T),\ \mu \in \sigma(S)\} .$$

It follows easily that 0 is an isolated point of $\sigma(C(T, S))$ if and only if $\sigma(T) \cap \sigma(S)$ consists of points which are isolated in both $\sigma(T)$ and $\sigma(S)$. When this occurs we shall say that T and S have *almost disjoint spectra*.

THEOREM 3.3. *If T and S are scalar operators in $\mathscr{B}(\mathscr{X})$, then $C(T, S)$ has closed range if and only if 0 is an isolated point of $\sigma(C(T, S))$. In particular $C(T, T)$ has closed range if and only if $\sigma(T)$ is finite.*

Proof. As remarked previously, $C(T, S)$ is a generalized scalar operator; hence a decomposable operator. Furthermore, by (1.5) its range meets its null space at an angle $\theta > 0$. Thus, if $\mathscr{R}(C(T, S))$ is closed 0 is an isolated point of $\sigma(C(T, S))$ by (3.1). Conversely, suppose that 0 is an isolated point of $\sigma(C(T, S))$. Then T and S have almost disjoint spectra. Let $\{\lambda_1, \cdots, \lambda_n\} = \sigma(T) \cap \sigma(S)$ and let $E(\cdot)$ and $F(\cdot)$ be the spectral resolutions of the identity associated with T and S respectively. Put $P_k = E(\{\lambda_k\})$, $Q_k = F(\{\lambda_k\})$ for $1 \leqq k \leqq n$, $P_0 = I - \sum_{k=1}^n P_k$, and $Q_0 = I - \sum_{k=1}^n Q_k$. Let $\mathscr{B}_{ij} = P_i \mathscr{B}(\mathscr{X}) Q_j$ for $0 \leqq i \leqq n$ and $0 \leqq j \leqq n$. Then each $\mathscr{B}_{ij}$ is an invariant subspace for $C(T, S)$ and the span of the $\mathscr{B}_{ij}$'s is $\mathscr{B}(\mathscr{X})$. Hence, it suffices to show that $C_{ij} = C(T, S)|_{\mathscr{B}_{ij}}$ has closed range for $0 \leqq i, j \leqq n$. Now if $i \neq 0$, $j \neq 0$ and $Y \in \mathscr{B}_{ij}$ then $C_{ij}(Y) = (\lambda_i - \lambda_j)Y$. Thus, in these cases C_{ij} has closed range. Also, if $1 \leqq i \leqq n$ and $Y \in \mathscr{B}_{j0}$, $C_{i0}(Y) = \lambda_i P_i Y - YS = \lambda_i P_i Y Q_0 - P_i Y Q_0 S = P_i Y Q_0(\lambda_i I - S)$. Since $(\lambda_i I - S)|_{Q_0\mathscr{X}}$ is invertible (λ_i is isolated in $\sigma(S)$) C_{i0} has closed range. Similarly, C_{0j} has closed range for $1 \leqq j \leqq n$. To complete the proof we show that C_{00} has closed range. Note that $\sigma(T|_{P_0\mathscr{X}}) \cap \sigma(S|_{Q_0\mathscr{X}}) = \varnothing$. Choose a real number k so that $k > \|S\| + \|T\|$ and define $S_1 = SP_0 + k(I - P_0)$ and $T_1 = Q_0 T - k(I - Q_0)$. Then $\sigma(S_1) \cap \sigma(T_1) = \varnothing$ and $C(T_1, S_1)$ is invertible on $\mathscr{B}(\mathscr{X})$. Thus, for each X in $\mathscr{B}(\mathscr{X})$ there is Y in $\mathscr{B}(\mathscr{X})$ so that $T_1 Y - YS_1 = X$. Hence, $P_0 X Q_0 = TP_0 Y Q_0 - P_0 Y Q_0 S$ and so C_{00} is onto $\mathscr{B}_{00}$.

4. In this section we restrict ourselves to scalar operators T and S acting on a complex infinite dimensional Hilbert space $\mathscr{H}$. Recall that an operator T in $\mathscr{B}(\mathscr{H})$ is of scalar type if and only if T is similar to a normal operator [10].

THEOREM 4.1. *Suppose that N_1 and N_2 are normal operators on a complex, infinite dimensional Hilbert space $\mathscr{H}$ and that $\sigma(N_1) \cap \sigma(N_2)$ contains a point λ which is an accumulation point for at least one of these spectra and is either an accumulation point or else an isolated eigenvalue of infinite multiplicity of the other. Then there is an operator V in $\mathscr{B}(\mathscr{H})$ such that the span of the null space and the range of $C(N_1, N_2)$ is orthogonal to the span of V. In particular $\mathscr{R}(C(N_1, N_2)) \dotplus \mathscr{N}(C(N_1, N_2))$ is not dense in $\mathscr{B}(\mathscr{H})$.*

Proof. We assume that $\lambda = \lim \lambda_n$ where $\{\lambda_n\}$ is a a sequence of distinct elements in $\sigma(N_1)$. The proof for the other case is similar. If λ is an accumulation point of $\sigma(N_2)$ choose a sequence $\{\mu_n\}$ of distinct elements in $\sigma(N_2)$ so that $\lambda = \lim \mu_n$. Taking subsequences if necessary, we may assume that the μ_n's are also distinct from the λ_n's. If λ is an isolated eigenvalue of infinite multiplicity for N_2 take $\lambda = \mu_n$.

Let $E_i(\cdot)$ be the spectral measure associated with N_i for $i = 1, 2$. Choose disks D_n of radius r_n about λ_n so that $D_n \cap D_m = \emptyset$ if $n \neq m$ and $\mu_n \notin D_m$ for any n or m. Note that since $\lambda_n \to \lambda$, $r_n \to 0$. For each n put $P_n = E_1(D_n)$, and choose a unit vector f_n in the range of P_n ($P_n \neq 0$ since $\lambda_n \in \sigma(N_1)$). If λ is an accumulation point of $\sigma(N_2)$ choose disks D'_n of radius r'_n about μ_n so that $D'_n \cap D'_m = \emptyset$ and $D'_n \cap D_n = \emptyset$ for all m and n. Put $Q_n = E_2(D'_n)$. If λ is an isolated eigenvalue of infinite multiplicity for N_2 let $\{Q_n\}$ be an infinite set of nonzero, mutually orthogonal projections such that $Q_n \leqq E_2(\{\lambda\})$ for each n. In either case for each n choose a unit vector e_n in the range of Q_n. Define V on $\mathscr{H}$ by $Ve_n = f_n$ and $Vx = 0$ for x in the orthogonal complement of the span of $\{Q_n\mathscr{H}\}_{n=1}^{\infty}$. Then $V \in \mathscr{B}(\mathscr{H})$ and $\|V\| = \|P_n V Q_n\| = 1$ for all n. In fact V is a partial isometry. Let X and W be in $\mathscr{B}(\mathscr{H})$ and suppose that $N_1 W = W N_2$. Then (as in (2.1)) $E_1(\delta) W = W E_2(\delta)$ for each Borel set $\delta \subset C$. Let

$$\alpha = \| V - W - (N_1 X - X N_2) \| .$$

To complete the proof we show that $\alpha \geqq 1$.

$$\begin{aligned}\alpha = \alpha \|P_n\| \, \|Q_n\| &\geqq \|P_n V Q_n - P_n W Q_n - (P_n N_1 X Q_n - P_n X N_2 Q_n)\| \\ &\geqq 1 - \|W E_2(D_n) Q_n\| - \|N_1 P_n X Q_n - P_n X Q_n N_2\| .\end{aligned}$$

Now $E_2(D_n)Q_n = 0$ since $D_n \cap D'_n = \emptyset$ and $\lambda \notin D_n$ so

$$\begin{aligned}\alpha \geqq 1 &- \|N_1 P_n X Q_n - \lambda_n P_n X Q_n\| - \|(\lambda_n - \mu_n) P_n X Q_n\| \\ &- \|\mu_n P_n X Q_n - P_n X Q_n N_2\| .\end{aligned}$$

Thus,

$$\begin{aligned}1 - \alpha &\leqq (\|N_1 P_n - \lambda_n P_n\| + |\lambda_n - \mu_n| + \|\mu_n Q_n - N_2 Q_n\|) \|P_n X Q_n\| \\ &\leqq (r_n + |\lambda_n - \mu_n| + r'_n) \|P_n X_n Q_n\|\end{aligned}$$

which goes to 0 as $n \to \infty$. Hence, $\alpha \geqq 1$.

THEOREM 4.2. *Let T and S be scalar operators in $\mathscr{B}(\mathscr{H})$. The following are equivalent.*

(a) *$C((T - \lambda I), S)$ has closed range for each $\lambda \in C$*

(b) *$C(T, S)$ is a spectral operator*

(c) *$C(T, S)$ is a scalar operator*

(d) *$\sigma(T) \cup \sigma(S)$ is finite.*

Proof. Clearly, $T - \lambda I$ and S have almost disjoint spectra for all λ in C if and only if $\sigma(T) \cup \sigma(S)$ is finite so (a) is equivalent to (d) by (3.3). If $\sigma(T) \cup \sigma(S)$ is finite then so is $\sigma(C(T, S))$. In this case the Riesz-Dunford theory implies that $C(T, S)$ is a spectral operator and, in virtue of (2.1) it is even scalar. Hence, (d) implies (b). Since the implication from (c) to (b) is trivial, to complete the proof we need only show that (b) implies (d). Hence, assume that $C(T, S)$ is a spectral operator. Since T and S are scalar operators in $\mathscr{B}(\mathscr{H})$, there are normal operators N_i and invertible operators X_i in $\mathscr{B}(\mathscr{H})$ for $i = 1, 2$ so that

$$T = X_1 N_1 X_1^{-1} \text{ and } S = X_2 N_2 X_2^{-1} .$$

Define Θ acting on $\mathscr{B}(\mathscr{H})$ by $\Theta(Y) = X_1^{-1} Y X_2$. Then

$$\Theta C(T, S)\Theta^{-1} = C(N_1, N_2)$$

and so $C(N_1, N_2)$ is a spectral operator. Let $E(\cdot)$ be the spectral resolution of the identity associated with $C(N_1, N_2)$. Then [3, p. 33]

$$B_C(\delta) = E(\delta)\mathscr{B}(\mathscr{H})$$

for all closed subsets δ in $\boldsymbol{C}$. Hence, by (2.1)

$$\mathscr{N}(C(N_1, N_2)) = \mathscr{B}_C(\{0\}) = E(\{0\})\mathscr{B}(\mathscr{H}) .$$

On the other hand, $\mathscr{B}_C(\delta) \subset \mathscr{R}(C(N_1, N_2))$ for each closed subset δ of $\boldsymbol{C}$ with $0 \notin \delta$ so that the countable additivity of $E(\cdot)$ implies

$$E(\{C\backslash\{0\}\})\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(C)^-$$

where the bar denotes the closure in the uniform topology. Therefore, the algebraic direct sum

$$\mathscr{R}(C)^- \dotplus \mathscr{N}(C) = \mathscr{B}(\mathscr{H}) .$$

Now $C_\lambda = C((N_1 - \lambda I), N_2) = C(N_1, N_2) - \lambda I$ is a spectral operator for each λ in $\boldsymbol{C}$ so as before

$$(*) \qquad \mathscr{R}(C_\lambda)^- + \mathscr{N}(C_\lambda) = \mathscr{B}(\mathscr{H})$$

for each complex λ. If both $\sigma(N_1)$ and $\sigma(N_2)$ were infinite then for some λ, $\sigma(N_1 - \lambda I) \cap \sigma(N_2)$ would contain a common accumulation point and by (4.1) $(*)$ would be false. Thus, either N_1 or N_2 has finite spectrum. Say $\sigma(N_2)$ is finite. Then, since $\mathscr{H}$ is infinite dimensional N_2 has an isolated eigenvalue of infinite multiplicity. If $\sigma(N_1)$ were infinite (4.1) would again contradict $(*)$. Thus, $\sigma(T) \cup \sigma(S) = \sigma(N_1) \cup \sigma(N_2)$ is finite.

Remarks 4.3. Clearly, (a) and (d) remain equivalent if T and S are scalar operators acting on a Banach space. We do not know, however, if (b) and (c) are also equivalent to (d) in this more general setting. Unfortunately, our proof does not appear to generalize. Our proof depends on the existence of an operator V which is not in the closed linear span of $\mathscr{R}(C)$ and $\mathscr{N}(C)$. Although V can be formally defined in the Banach space setting, it is not clear that V will be bounded.

5. In this section we investigate the extent to which properties (F) and (G) hold for $C(T, S)$ and use our results to give an example of a Hermitian operator whose square is not Hermitian. We begin by recalling the definition of the numerical range of an element in a Banach algebra.

DEFINITION 5.1. Let $\mathfrak{A}$ be a complex Banach algebra with identity I. The set of *states* on $\mathfrak{A}$ is by definition

$$\mathscr{P} = \{f \in \mathfrak{A}^*: f(I) = 1 = \|f\|\} .$$

The numerical range of an element a in $\mathfrak{A}$ is by definition the set

$$W_0(a) = \{f(a): f \in \mathscr{P}\} .$$

Since $\mathscr{P}$ is a weak* closed convex subset of the unit ball of $\mathfrak{A}^*$, $W_0(a)$ is a closed convex subset of $\boldsymbol{C}$ for each $a \in \mathfrak{A}$. If $\mathfrak{A} = \mathscr{B}(\mathscr{H})$ $W_0(\cdot)$ is just the closure of $W(\cdot)$ the usual numerical range (for further information see [9]). The *numerical radius* of a is by definition $\sup\{|\lambda|: \lambda \in W_0(a)\}$. The *spectral radius* of a is by definition $\sup\{|\lambda|: \lambda \in \sigma(a)\}$. An element a in $\mathfrak{A}$ is *Hermitian* if $W_0(a)$ is real. Recall [7] that a in $\mathfrak{A}$ is Hermitian if and only if $\|\exp(ita)\| = 1$ for all t in $\boldsymbol{R}$. For an operator T in $\mathscr{B}(\mathscr{X})$ define the operators L_T and R_T in $\mathscr{B}(\mathscr{B}(\mathscr{X}))$ by $L_T(X) = TX$ and $R_T(X) = XT$.

THEOREM 5.2. *If T is in $\mathscr{B}(\mathscr{X})$ then $W_0(T) = W_0(L_T) = W_0(R_T)$.*

Proof. For each state f on $\mathscr{B}(\mathscr{B}(\mathscr{X}))$ the formula $g(X) = f(L_X)$ determines a state g on $\mathscr{B}(\mathscr{X})$. Hence, $W_0(L_T) \subset W_0(T)$. Conversely for each state f on $\mathscr{B}(\mathscr{X})$ the formula $g(L_X) = f(X)$ determines a state g on $\{L_X: X \in \mathscr{B}(\mathscr{X})\}$ which then extends by the Hahn-Banach theorem to a state on all of $\mathscr{B}(\mathscr{B}(\mathscr{X}))$. It follows that $W_0(T) \subset W_0(L_T)$ and so $W_0(T) = W_0(L_T)$. Similarly, $W_0(T) = W_0(R_T)$.

COROLLARY 5.3. *If T and S are Hermitian operator in $\mathscr{B}(\mathscr{X})$, then $C(T, S)$ is a Hermitian operator.*

Proof. $C(T, S) = L_T - R_S$. Thus

$$W_0(C(T, S)) \subset W_0(L_T) - W_0(R_S) = W_0(T) - W_0(S) \subset R .$$

THEOREM 5.4. *If N_1 and N_2 are normal operators in $\mathscr{B}(\mathscr{H})$ where $\mathscr{H}$ is a complex Hilbert space then $C(N_1, N_2) = H + iJ$ where H and J are commuting Hermitian operators.*

Proof. Write $N_j = A_j + iB_j$, where A_j and B_j are the real and imaginary parts of N_j for $j = 1, 2$. Let $H = C(A_1, A_2)$ and $J = C(B_1, B_2)$. Then H and J are Hermitian operators by (5.3) and since $A_jB_j = B_jA_j$ for $j = 1, 2$, H and J commute.

THEOREM 5.5. *If N_1 and N_2 are normal operators on a complex Hilbert space then the spectral radius and the numerical radius of $C(N_1, N_2)$ are equal.*

Proof. Palmer has shown [7, Lemma 1.6] that the conclusion of the theorem holds for all operators of the form $H + iJ$ when H and J are commuting Hermitian operators. Hence, the theorem follows from (5.4).

EXAMPLE 5.6. The norm and the spectral radius of $C(N_1, N_2)$ need not be equal, however. Indeed, Stampfli has shown [8] that

$$\| C(N_1, N_2) \| = \inf_{\lambda \in C} \{\| N_1 - \lambda I\| + \| N_2 - \lambda I \|\} .$$

Thus, if N is a normal operator in $\mathscr{B}(\mathscr{H})$ whose spectrum is an equilateral triangle of side 1 then (Lumer and Rosenblum) the spectral radius of $\Delta_N = C(N, N)$ is 1 but since the norm of $N - \lambda I$ is equal to the spectral radius of $N - \lambda I$, $\| \Delta_N \| = 2/\sqrt{3}$. Note that in this case $\sigma(\Delta_N)$ is a solid hexagon centered at the origin. On the other hand, $C(N_1, N_2)$ is a convexoid operator.

THEOREM 5.7. *If N_1 and N_2 are normal ope on a complex Hilbert space then the convex hull of $\sigma(C($ is equal to $W_0(C(N_1, N_2))$.*

Proof. Let conv $(\cdot)$ denote the convex hull of the set within the parentheses. Since $\sigma(a) \subset W_0(a)$ for any element a of a Banach algebra $\mathfrak{A}$ [9, Theorem 1] it suffices to show $W_0(C(N_1, N_2)) \subset \operatorname{conv}(\sigma(C(N_1, N_2)))$. Now

$$\begin{aligned} W_0(C(N_1, N_2)) \subset W_0(N_1) - W_0(N_2) &= W(N_1)^- - W(N_2)^- \\ &= \operatorname{conv}(\sigma(N_1)) - \operatorname{conv}(\sigma(N_2)) \\ &= \operatorname{conv}(\sigma(N_1) - \sigma(N_2)) = \operatorname{conv}(\sigma(C(N_1, N_2))) . \end{aligned}$$

where we have used (5.2) and property (G) of the introduction.

THEOREM 5.8. *If P is a* (self-adjoint) *projection on a complex Hilbert space $\mathscr{H}$ then $L_P R_P$ is a Hermitian operator if and only if P is* 0 *or* I.

Proof. We give the proof in the case that $\mathscr{H}$ has dimension 2 and $P\mathscr{H}$ has dimension 1. The generalization to higher dimensions is obvious. As remarked in (5.1) it suffices to show that $\|\exp(it\ L_P R_P)\| > 1$ for some t in $\boldsymbol{R}$. Let

$$T = \begin{bmatrix} (1/2)i & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

on $P\mathscr{H} \oplus (I - P)\mathscr{H}$ and let $t = 3\pi/2$. Then since $\exp(it\ L_P R_P) = I + (e^{it} - 1)L_P R_P$,

$$\exp(it\ L_P R_P)(T) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

which is a projection and, hence, has norm 1. On the other hand, $\|T^* T\|^2 < 3/4$ as an easy computation shows. Hence, $\|\exp(it\ L_P R_P)\| > 1$ for $t = 3\pi/2$ and $L_P R_P$ is not Hermitian.

EXAMPLE 5.9. Let P be a projection as in (5.8). Then Δ_P is a Hermitian operator by (5.3) but Δ_P^2 is not a Hermitian oprator. Indeed, since the Hermitian operators on $\mathscr{B}(\mathscr{H})$ form a real vector space and

$$L_P R_P = \frac{1}{2}(\Delta_P - \Delta_P^2)$$

it follows from (5.8) that Δ_P^2 is not Hermitian.

In conclusion we remark that Crabbe [4] and Browder [2] have given similar examples.

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